

Linear Hilbertian manifold domains

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Abstract

Necessary and sufficient conditions for a dense subspace of a Hilbert space to be a linear Hilbertian manifold domain are given. Some relations between linear Hilbertian manifold domains and domains of self-adjoint operators are found.

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1 Introduction

Manifold domains play an important role in geometric formulation of quantum mechanics [1, 2] and in general infinite dimensional Hamiltonian systems [3].

Definition 1.1. *A subset D of a Banach manifold M is a manifold domain provided*

1. *D is dense in M ,*
2. *D carries a Banach manifold structure of its own such that the inclusion $i : D \rightarrow M$ is smooth,*
3. *for each $p \in D$, the linear map $T_p i : T_p D \rightarrow T_p M$ is a dense inclusion.*

Note that D is not a submanifold of M in the sense of S.Lang [4]. In this paper we consider the case when M is a Hilbert space, D is a dense linear subspace of M and D carries a Hilbert space structure of its own. Then it is obvious that $i : D \rightarrow M$ is linear and if $i : D \rightarrow M$ is continuous then it is also holomorphic if M is complex and smooth if M is real. Thus we arrive at the following

Definition 1.2. *A linear subspace D of a Hilbert space H is called a linear Hilbertian manifold domain of H if*

1. *D is dense in H ,*
2. *D carries a Hilbert space structure of its own such that the inclusion $i : D \rightarrow H$ is continuous.*

(Observe that in the linear case the assumption (3) of Definition 1.1 is contained in (1) and (2)).

Perhaps, the most transparent example of a linear Hilbertian manifold domain is the Sobolev space $W_2^m(\Omega)$. Let Ω be some domain of \mathbb{R}^n and $L^2(\Omega)$ be the Hilbert space of all complex functions on Ω for which $|\cdot|^2$ is integrable on Ω . The scalar product and the norm in $L^2(\Omega)$ are defined by

$$(f, g)_{L^2(\Omega)} := \int_{\Omega} f(x) \overline{g(x)} dx \quad (1)$$

$$\|f\|_{L^2(\Omega)} := \sqrt{(f, f)_{L^2(\Omega)}}, \quad f, g \in L^2(\Omega) \quad (2)$$

(The overbar stands for the complex conjugation). Sobolev space $W_2^m(\Omega)$ is defined by

$$W_2^m(\Omega) := \{f \in L^2(\Omega) : D^{(i_1, \dots, i_n)} f \in L^2(\Omega), 0 \leq i_1 + \dots + i_n \leq m\} \quad (3)$$

where $D^{(i_1, \dots, i_n)} f := \frac{\partial^{i_1 + \dots + i_n} f}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}$ and all derivatives are taken in the sense of the theory of distributions. Then one introduces a scalar product $(\cdot, \cdot)_{W_2^m(\Omega)}$

$$(u, v)_{W_2^m(\Omega)} := \sum_{0 \leq i_1 + \dots + i_n \leq m} (D^{(i_1, \dots, i_n)} u, D^{(i_1, \dots, i_n)} v)_{L^2(\Omega)} \quad (4)$$

for $u, v \in W_2^m(\Omega)$. It can be shown that $W_2^m(\Omega)$ endowed with the scalar product (4) is a separable Hilbert space. It is also obvious that

$$\|u\|_{L^2(\Omega)} \leq \|u\|_{W_2^m(\Omega)} \quad (5)$$

and this leads to the conclusion that the inclusion $i : W_2^m(\Omega) \rightarrow L^2(\Omega)$ of the Hilbert space $(W_2^m(\Omega), (\cdot, \cdot)_{W_2^m(\Omega)})$ into $L^2(\Omega)$ is continuous. Finally, since $W_2^m(\Omega)$ is a dense linear subspace of $L^2(\Omega)$ one concludes that the Sobolev space $W_2^m(\Omega)$ is a linear Hilbertian manifold domain of $L^2(\Omega)$. To follow further we consider a Hilbert space \tilde{H} ,

$$\tilde{H} := \{(f^{(0, \dots, 0)}, \dots, f^{(i_1, \dots, i_n)}, \dots, f^{(0, \dots, m)}) : 0 \leq i_1 + \dots + i_n \leq m, f^{(i_1, \dots, i_n)} \in L^2(\Omega)\} \quad (6)$$

$$(F, G)_{\tilde{H}} := \sum_{0 \leq i_1 + \dots + i_n \leq m} \left(f^{(i_1, \dots, i_n)}, g^{(i_1, \dots, i_n)} \right)_{L^2(\Omega)} \quad (7)$$

where $F = (\dots, f^{(i_1, \dots, i_n)}, \dots)$, $G = (\dots, g^{(i_1, \dots, i_n)}, \dots) \in \tilde{H}$. Then one defines a linear operator $D^{(m)} : W_2^m(\Omega) \rightarrow \tilde{H}$

$$D^{(m)}u := \left(D^{(0, \dots, 0)}u, \dots, D^{(i_1, \dots, i_n)}u, \dots \right), \quad 0 \leq i_1 + \dots + i_n \leq m, \quad u \in W_2^m(\Omega). \quad (8)$$

It can be shown that the operator $D^{(m)}$ is closed. With the use of $D^{(m)}$ the scalar product (4) in W_2^m reads

$$(u, v)_{W_2^m(\Omega)} = (D^{(m)}u, D^{(m)}v)_{\tilde{H}}, \quad u, v \in W_2^m(\Omega). \quad (9)$$

Now the natural question arises if for any linear Hilbertian manifold domain $D \subset H$ there exists a closed linear operator A from D to some Hilbert space \tilde{H} such that the Hilbert structure on D is defined analogously to (9) by

$$(\psi, \phi)_D = (A\psi, A\phi)_{\tilde{H}}, \quad \psi, \phi \in D. \quad (10)$$

The answer to this question is the main point of our paper.

2 Main theorem

We start with the following

Lemma 2.1. *Let H be a Hilbert space and $D \subset H$ be a linear dense domain of n closed linear operators: $A_1 : D \rightarrow H_1, \dots, A_n : D \rightarrow H_n$ where H_1, \dots, H_n are Hilbert spaces. For $1 \leq k \leq n$ define a scalar product in D by*

$$(\psi, \phi)_D^{(k)} := (\psi, \phi)_H + \sum_{i=1}^k (A_i\psi, A_i\phi)_{H_i}, \quad \psi, \phi \in D. \quad (11)$$

Then $(D, (\cdot, \cdot)_D^{(k)})$ is a Hilbert space and the inclusion $i : D \rightarrow H$ of $(D, (\cdot, \cdot)_D^{(k)})$ into $(H, (\cdot, \cdot)_H)$ is continuous, what implies that D is a linear Hilbertian manifold domain of H . Moreover, the norms $\|\cdot\|_D^{(k_1)} = \sqrt{(\cdot, \cdot)_D^{(k_1)}}$ and $\|\cdot\|_D^{(k_2)} = \sqrt{(\cdot, \cdot)_D^{(k_2)}}$ are equivalent for any $1 \leq k_1, k_2 \leq n$.

Proof. Let $\{\phi_j\}_1^\infty$, $\phi_j \in D$, be a Cauchy series with respect to the norm $\|\cdot\|_D^{(k)}$ ($1 \leq k \leq n$), i.e.,

$$\lim_{j,l \rightarrow \infty} \|\phi_j - \phi_l\|_D^{(k)} = 0. \quad (12)$$

From (11) and the definition of $\|\cdot\|_D^{(k)}$ it follows that (12) yields

$$\lim_{j,l \rightarrow \infty} \|\phi_j - \phi_l\|_H = 0 \quad (13)$$

$$\lim_{j,l \rightarrow \infty} \|A_i \phi_j - A_i \phi_l\|_{H_i} = 0, \quad i = 1, \dots, k \quad (14)$$

what means that $\{\phi_j\}_1^\infty, \{A_1 \phi_j\}_1^\infty, \dots, \{A_k \phi_j\}_1^\infty$ are Cauchy series in H, H_1, \dots, H_k respectively. Hence, there exist $\phi \in H, \psi_1 \in H_1, \dots, \psi_k \in H_k$ such that

$$\lim_{j \rightarrow \infty} \|\phi_j - \phi\|_H = 0 \quad (15)$$

$$\lim_{j \rightarrow \infty} \|A_i \phi_j - \psi_i\|_{H_i} = 0, \quad i = 1, \dots, k. \quad (16)$$

Since A_i, \dots, A_k are closed

$$\phi \in D \quad (17)$$

and

$$A_i \phi = \psi_i, \quad i = 1, \dots, k. \quad (18)$$

Therefore

$$\left(\|\phi_j - \phi\|_D^{(k)} \right)^2 = \|\phi_j - \phi\|_H^2 + \sum_{i=1}^k \|A_i(\phi_j - \phi)\|_{H_i}^2 \rightarrow 0 \quad (19)$$

and it means that $(D, \|\cdot\|_D^{(k)})$ is complete for each $1 \leq k \leq n$. Consequently $(D, (\cdot, \cdot)_D^{(k)})$ is a Hilbert space for any $1 \leq k \leq n$. Obviously

$$\|\psi\|_H \leq \|\psi\|_D^{(k)}, \quad \psi \in D, \quad k = 1, \dots, n. \quad (20)$$

From (20) one gets that the inclusion $i : D \rightarrow H$ is a continuous mapping from $(D, (\cdot, \cdot)_D^{(k)})$ into $(H, (\cdot, \cdot)_H)$. Thus D is a linear Hilbertian manifold domain of H . Moreover

$$\|\cdot\|_D^{(k_1)} \leq \|\cdot\|_D^{(k_2)} \quad (21)$$

for any $k_1 \leq k_2$. Therefore, employing *the open mapping theorem* [5] one concludes that the norms $\|\psi\|_D^{(k_1)}$ and $\|\psi\|_D^{(k_2)}$ are equivalent. The proof is complete. \square

Observe that defining

$$\tilde{H}^{(k)} = H \oplus H_1 \oplus \dots \oplus H_k$$

$$\tilde{A}^{(k)} : D \rightarrow \tilde{H}^{(k)}, \quad \tilde{A}^{(k)}(\psi) := \psi \oplus A_1 \psi \oplus \dots \oplus A_k \psi, \quad \psi \in D, \quad 1 \leq k \leq n. \quad (22)$$

we can rewrite (11) in the following compact form

$$(\psi, \phi)_D^{(k)} = \left(\tilde{A}^{(k)} \psi, \tilde{A}^{(k)} \phi \right)_{\tilde{H}^{(k)}}. \quad (23)$$

Of course $\tilde{A}^{(k)} : D \rightarrow \tilde{H}^{(k)}$ is a closed linear operator. From Lemma 2.1 one easily obtains

Corollary 2.1. [1] *If $D_A \subset H$ is a dense domain of a self-adjoint operator $A : D_A \rightarrow H$, $A = A^*$, then $(D_A, (\cdot, \cdot)_{D_A})$ with*

$$(\phi, \psi)_{D_A} := (\phi, \psi)_H + (A\phi, A\psi)_H \quad (24)$$

is a Hilbert space and the inclusion $i : D \rightarrow H$ is continuous, what implies that D_A is a linear Hilbertian manifold domain of H .

Proof. $A = A^* \implies A$ is closed. Then from Lemma 2.1 we get the thesis. \square

The fact that any dense domain of a self-adjoint operator is a linear Hilbertian manifold domain appears to be fundamental in geometrical formulation of quantum mechanics (see [1]). Before giving the main theorem we prove a simple lemma

Lemma 2.2. *Let $(H, (\cdot, \cdot)_H)$ be a Hilbert space and $D \subset H$ be a dense linear subspace of H . If there exists a scalar product $(\cdot, \cdot)_D$ in D such that $(D, (\cdot, \cdot)_D)$ is a separable Hilbert space and $\|f\|_H \leq a\|f\|_D \ \forall f \in D$, $a > 0$, then*

$$\dim(H, (\cdot, \cdot)_H) = \dim(D, (\cdot, \cdot)_D). \quad (25)$$

Proof. For $\dim(D, (\cdot, \cdot)_D) < \infty$ the proof is trivial. Let $\dim(D, (\cdot, \cdot)_D) = \aleph_0$, where \aleph_0 is the cardinal number of the set of natural numbers \mathbb{N} . From the assumption that the norm $\|\cdot\|_D$ is stronger than $\|\cdot\|_H$ and D is dense in $(H, (\cdot, \cdot)_H)$ it follows that every set which is dense in $(D, (\cdot, \cdot)_D)$ is also dense in $(H, (\cdot, \cdot)_H)$. Since $(D, (\cdot, \cdot)_D)$ is separable there exists a countable dense set S in $(D, (\cdot, \cdot)_D)$. Then S is also dense in $(H, (\cdot, \cdot)_H)$. Thus $(H, (\cdot, \cdot)_H)$ is separable and (25) holds true. \square

From Lemma 2.2 one gets

Corollary 2.2. *If a linear Hilbertian manifold domain D of H admits a scalar product $(\cdot, \cdot)_D$ such that $(D, (\cdot, \cdot)_D)$ is a separable Hilbert space and $\|f\|_H \leq \|f\|_D$ then the Hilbert space $(H, (\cdot, \cdot)_H)$ is also separable.* \square

Now we are in a good position to prove the main theorem:

Theorem 2.1. *Let $(H, (\cdot, \cdot)_H)$ be a Hilbert space and let $D \subset H$ be a dense linear subspace of H . If there exists a linear operator $A : D \rightarrow H$ satisfying the following conditions*

- (a) *A is one-to-one and $A(D) = H$,*
- (b) *The inverse operator $A^{-1} : H \rightarrow D$ is bounded,*

then D is a linear Hilbertian manifold domain of H and

$$(f, g)_D := (A \circ if, A \circ ig)_H, \quad f, g \in D, \quad (26)$$

where $i : D \rightarrow H$ is the inclusion of D into H , defines a scalar product in D such that $(D, (\cdot, \cdot)_D)$ is a Hilbert space and the inclusion $i : D \rightarrow H$ of the Hilbert space $(D, (\cdot, \cdot)_D)$ into $(H, (\cdot, \cdot)_H)$ is continuous.

Conversely, if $D \subset H$ is a linear Hilbertian manifold domain of H and $(\cdot, \cdot)_D$ is any scalar product in D such that $(D, (\cdot, \cdot)_D)$ is a separable Hilbert space and the inclusion $i : D \rightarrow H$ is continuous then there exists a linear operator $A : D \rightarrow H$ satisfying (a) and (b) and such that the scalar product $(\cdot, \cdot)_D$ is given by (26). Any linear operator satisfying (a) and (b) is closed.

Proof. Let $A : D \rightarrow H$ be a linear operator fulfilling (a) and (b). One easily shows that the formula (26) defines a scalar product $(\cdot, \cdot)_D$ in D . We prove that $(D, (\cdot, \cdot)_D)$ is a Hilbert space. Let $\{f_n\}_1^\infty$, $f_n \in D$, be a Cauchy series in $(D, (\cdot, \cdot)_D)$ i.e.,

$$\lim_{n, m \rightarrow \infty} \|f_n - f_m\|_D = 0. \quad (27)$$

Denote $h_n \ni h_n := A \circ if_n$. By (26) and (27)

$$\lim_{n, m \rightarrow \infty} \|h_n - h_m\|_H = 0.$$

This means that $\{h_n\}_1^\infty$ is a Cauchy series in H and, consequently, there exists $h \in H$ such that

$$\lim_{n, m \rightarrow \infty} \|h_n - h\|_H = 0.$$

Define $f := i^{-1} \circ A^{-1}h \in D$. Then

$$\|f_n - f\|_D = \|h_n - h\|_H \rightarrow 0.$$

Hence $f \in D$ is the limit of the Cauchy series $\{f_n\}_1^\infty$. This proves the completeness of $(D, (\cdot, \cdot)_D)$ and one concludes that $(D, (\cdot, \cdot)_D)$ is a Hilbert space. For any vector g of this Hilbert space

$$\|g\|_D = \|A \circ ig\|_H. \quad (28)$$

By the condition (b)

$$\|ig\|_H = \|A^{-1} \circ A \circ ig\|_H \leq \|A^{-1}\| \|A \circ ig\|_H \Rightarrow \|ig\|_H \leq \|A^{-1}\| \|g\|_D. \quad (29)$$

Therefore, the norm $\|\cdot\|_D$ is stronger than $\|\cdot\|_H$ and the inclusion $i : D \rightarrow H$ is continuous. This ends the proof of the first part of our theorem.

To prove the second part, first note that under the assumption that the Hilbert space $(D, (\cdot, \cdot)_D)$ is separable Lemma 2.2 gives

$$\dim(D, (\cdot, \cdot)_D) = \dim(H, (\cdot, \cdot)_H) = \aleph_0.$$

Hence there exists an isometry $U : H \rightarrow D$ of $(H, (\cdot, \cdot)_H)$ onto $(D, (\cdot, \cdot)_D)$ [5, 6]

$$(Uh_1, Uh_2)_D = (h_1, h_2)_H, \quad h_1, h_2 \in H. \quad (30)$$

Then the linear operator $i \circ U : H \rightarrow D \subset H$ is a one-to-one bounded (with respect to $\|\cdot\|_H$) linear operator from H onto D . Consequently, the inverse operator

$$(i \circ U)^{-1} =: A : D \rightarrow H$$

is a one-to-one linear operator from D onto H and obviously it satisfies the conditions (a) and (b). Then it is easy to see that (30) leads to (26). This proves the second part of the theorem.

Finally, since $A^{-1} : H \rightarrow D$ is by (b) a bounded linear operator, it is also closed. Hence, $A : D \rightarrow H$ being an inverse operator to the closed linear operator A^{-1} is also closed [6]. The proof is complete. \square

The formula (26) corresponds to the formulas (9), (10) or (23) but now the Hilbert spaces \tilde{H} or $\tilde{H}^{(k)}$ are exactly the original Hilbert space H to which the linear Hilbertian manifold domain D belongs. From Theorem 2.1 one concludes that:

Every linear Hilbertian manifold domain D of H , admitting a scalar product $(\cdot, \cdot)_D : D \times D \rightarrow \mathbb{C}$ such that $(D, (\cdot, \cdot)_D)$ is a separable Hilbert space and the inclusion $i : D \rightarrow H$ is continuous, is the domain of some linear operator $A : D \rightarrow H$ satisfying the conditions (a) and (b). These conditions imply that A is closed.

Corollary 2.1 tells us that any dense domain of a self-adjoint operator is a linear Hilbertian manifold domain. We are not able to prove if the inverse statement is also true. Nevertheless, in the next section we prove some weakened form of this statement.

3 Self-adjoint operators and linear Hilbertian manifold domains

Let H be a separable Hilbert space and $D \subset H$ be a linear Hilbertian manifold domain of H . Employing Theorem 2.1 one can easily show that there exists a linear Hilbertian manifold domain D_A of H , $D_A \subset D \subset H$, and an operator $A : D_A \rightarrow H$ satisfying (a) and (b). Since A is a closed linear operator and D_A is dense in H the operator $B := A^*A$ is self-adjoint and positive, and the domain of B , $D_B \subset D_A \subset D \subset H$ is dense in H (see [6]). $A^{-1} : H \rightarrow D_A$ is a one-to-one and bounded linear operator from H onto D_A which implies that $(A^{-1})^* : H \rightarrow D_A^*$ is also a one-to-one, bounded linear operator from H onto D_A^* . It can be proved [6] that $(A^{-1})^* = (A^*)^{-1}$. Consequently, the linear operator $B = A^*A : D_B \rightarrow H$ is a one-to-one, self adjoint and positive operator from D_B onto $B(D_B)$. We define scalar product $(\cdot, \cdot)_B$ in D_B by

$$(f, g)_{D_B} := (Bf, Bg)_H, \quad f, g \in D_B \quad (31)$$

One can show that

$$\|f\|_{D_B} \geq \frac{1}{\|A^{-1}\|^2} \|f\|_H, \quad f \in D_B \quad (32)$$

where as usually $\|f\|_{D_B} = \sqrt{(f, f)_{D_B}}$.

Let $\{f_n\}_1^\infty$, $f_n \in D_B$, be a Cauchy series in $(D_B, (\cdot, \cdot)_{D_B})$ i.e.,

$$\lim_{n, m \rightarrow \infty} \|f_n - f_m\|_{D_B} = 0 \quad (33)$$

By (32) and (33) we get

$$\lim_{n,m \rightarrow \infty} \|f_n - f_m\|_H = 0 \quad (34)$$

Hence, there exists $f \in H$ such that

$$\lim_{n \rightarrow \infty} \|f_n - f\|_H = 0 \quad (35)$$

But from (31) and (33) it follows that there exists $g \in H$ such that

$$\lim_{n \rightarrow \infty} \|Bf_n - g\|_H = 0 \quad (36)$$

Since $B : D_B \rightarrow H$ is self-adjoint it is also closed and, consequently, $f \in D_B$ and $g = Bf$. Finally, one concludes that

$$\lim_{n \rightarrow \infty} \|f_n - g\|_{D_B} = \lim_{n \rightarrow \infty} \|Bf_n - Bf\|_H = 0 \quad (37)$$

and it means that $(D_B, (\cdot, \cdot)_{D_B})$ is a Hilbert space. Moreover, by (32) the inclusion $i : D_B \rightarrow H$ is continuous. Thus we arrive at the following

Theorem 3.1. *Let D be a linear Hilbertian manifold domain of a separable Hilbert space $(H, (\cdot, \cdot)_H)$. Then there exists a dense linear subspace D_B of H , $D_B \subset D \subset H$ and a self-adjoint positive linear operator $B : D_B \rightarrow H$ such that $(D_B, (\cdot, \cdot)_{D_B})$ is a Hilbert space with*

$$(f, g)_{D_B} := (Bf, Bg)_H, \quad f, g \in D_B$$

and the inclusion $i : D_B \rightarrow H$ is a continuous inclusion of $(D_B, (\cdot, \cdot)_{D_B})$ into $(H, (\cdot, \cdot)_H)$. \square

Analogously as in Lemma 2.1 one gets that the scalar product $\langle \cdot, \cdot \rangle_{D_B}$ in D_B defined by

$$\langle f, g \rangle_{D_B} = (f, g)_H + (Bf, Bg)_H, \quad f, g \in D_B \quad (38)$$

leads to the norm in D_B equivalent to that given by the scalar product $(\cdot, \cdot)_{D_B}$.

Concluding one can say that:

If D is a linear Hilbertian manifold domain of a separable Hilbert space H then there exists a linear Hilbertian manifold domain $D_B \subset D \subset H$ of H which is the domain of some self-adjoint positive linear operator $B : D_B \rightarrow H$.

This conclusion is a weakened form of the statement inverse to Corollary 2.1.

4 Manifold domains and observables in geometric quantum mechanics

In ordinary quantum mechanics observables are represented by densely defined self-adjoint linear operators in a separable Hilbert space H . As is well known if $\mathcal{O} : D_{\mathcal{O}} \rightarrow H$ is a self-adjoint linear operator from $D_{\mathcal{O}} \subset H$ to H then

$$\exp\{it\mathcal{O}\} := \int_{-\infty}^{+\infty} \exp\{it\lambda\} dE_{\lambda}, \quad t \in (-\infty, +\infty) \quad (39)$$

(where E_{λ} , $\lambda \in (-\infty, +\infty)$ is the spectral measure associated with \mathcal{O}) defines a strongly continuous one-parameter group of unitary operators on H . Conversely, by

Stone's theorem, every strongly continuous one-parameter group of unitary operators on H is given by (39). Hence, in quantum mechanics observables be eventually identified with strongly continuous one-parameter groups of unitary operators on H . In geometric quantum mechanics [1, 2, 7, 8, 9, 10] an observable is represented by the so called *observable function*

$$H \supset D_{\mathcal{O}} \ni f \rightarrow \langle \mathcal{O} \rangle(f) := (\mathcal{O}f, f)_H \in \mathbb{R}. \quad (40)$$

From Corollary 2.1 it follows that $D_{\mathcal{O}}$ is a linear Hilbertian manifold domain of H . Then the Hamiltonian vector field X on $D_{\mathcal{O}}$ defined by the observable function (40)

$$X_f \lrcorner \omega = d\langle \mathcal{O} \rangle(f), \quad f \in D_{\mathcal{O}} \quad (41)$$

where ω is the natural symplectic form on H

$$\omega(f, g) := -2\text{Im}(f, g), \quad f, g \in H \quad (42)$$

generates (uniquely) the one-parameter continuous group $\mathbb{R} \times H \ni (t, g) \mapsto \varphi_t(g) \in H$ of Kähler isomorphisms of H satisfying

$$\left. \frac{d\varphi_t(f)}{dt} \right|_{t=0} = X_f, \quad f \in D_{\mathcal{O}}. \quad (43)$$

(Recall that a Kähler isomorphism of H is such an isomorphism which preserves the Kähler structure on H)

Conversely, again by Stone's theorem, every one-parameter continuous group $\mathbb{R} \times H \ni (t, g) \mapsto \varphi_t(g) \in H$ of Kähler isomorphisms of H defines (uniquely) an observable function $\langle \mathcal{O} \rangle$ determined on some linear Hilbertian manifold domain $D_{\mathcal{O}} \subset H$ according to (40). The Hamiltonian vector field X on $D_{\mathcal{O}}$ given by (41) satisfies the relation (43).

Remark: All that can be easily carry over to the projective space PH which is the true phase space in quantum theory (see [1, 2] and [7, 8, 9, 10] for details). We don't consider this problem as we deal with linear domains of Hilbert space H .

From Theorem 3.1 one can extract an interesting conclusion:

For any linear Hilbertian manifold domain D of a separable Hilbert space H there exists a positive observable function $\langle B \rangle : D_B \rightarrow \mathbb{R}$ ($\langle B \rangle(f) \geq 0 \forall f \in D_B$; $\langle B \rangle(f) = 0 \iff f = 0$) where $D_B \subset D \subset H$ is some linear Hilbertian manifold domain.

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